Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4, No.14, 2014
www.iiste.org

Some Types of Ideals on KS-Semigroups

Sajda mohammed[1*] Sundus Jaber[2]

1. Faculty of Education For Girls , Kufa university, Iraq

2. Faculty of Education For Girls , Kufa university, Iraq

*E-mail of the corresponding author Sajidak.mohammed@uokufa.edu.iq

Abstract:

In this paper we introduce a new types of ideals in KS- Semigroups in ordinary and fuzzy sense,we called it KS-H- ideal and fuzzy KS-H-ideal and study its properties

1.Introduction

The notation of BCK algebra introduced by Y.Imai and K.Ise'ki [3] in 1966 . In the same year , K.Ise'ki [2]introduced the notation of BCI algebra which is a generalization of BCK algebra. In 2006 ,Kyung Ho Kim

[5] introduced a new class of algebraic structure called KS semigroup .In 2009 Jocelyns S. Paradero Vilea and Mila Cawi [10] characterized ideals of KS- Semigroups and prove some properties .In 2007 , D.R. Prince Wiliams and Husain Shamshad[9] fuzzify KS semigroup and called it fuzzy KS Semigroups and introduced the notations of fuzzy subKS- Semigroups,,fuzzy KS ideal ,fuzzy KS P ideal and investigated some of their related properties in this paper we define a KS –H ideal and a fuzzy KS H- ideal on KS –Semigroups , we prove some of properties on it .

keywords: Semigroup, BCK algebra, H-ideal, P-ideal,ideal, Ks –semigroup,

2.Preliminary

This section contains some basic concepts we needed it in this paper

Definition (2.1)[9]: An algebraic system (X, $*$, 0) is called a ***BCK algebra*** if it satisfies the following conditions:

1. $((x*y)*(x*z))*(z*y)=0$,

2. $((x*(x*y))*y=0$,

3. $x*x=0$,

4. $0*x=0$

5.If $x*y=0$ *and* $y*x=0$ *then* $x=y$, *for all* $x, y, z \in X$.

Remarks (2.2)[6] : Let X be a BCK algebra then:

a) A partial ordering " $\le$ " on X can be defined by $x \le y$ if and only if $x*y=0$.

b) A BCK-algebra X has the following properties:

1. $x*0=x$.

2.If x*y=0 and y*z=0 imply x*z=0 .

3.If x*y=0 implies (x*z)*(y*z)=0 and (z*y)*(z*x)=0 .

4. If (x*y)*z=(x*z)*y.

Definition (2. 3)[9]

A ***KS-semigroup*** is a non-empty set X with two binary operation " $*$ " and " **.**" , and a constant **0** satisfies the following axioms:

1. (X, $*$, 0) is a *BCK-algebra*

2. (X, .) is a *semigroup*,

3. $x.(y * z) = (x.y) * (x.z)$ *and* $(x * y).z = (x.z) * (y.z)$, for all $x, y, z \in X$.

Definition (2. 4) [9] A non empty subset S of X with binary operation * and **.** is called ***sub KS-semigroup*** of X if it satisfies the following condition :

1- $x*y \in S \quad \forall \ x, y \in S$.

2- $x.y \in S \quad \forall \quad x, y \in S$

Definition (2. 5) [7] A ***strong KS-semigroup*** is a KS-semigroup X satisfying : $x*y=x*x.y$ *for all* $x, y \in X$

Lemma(2.6) [7]:Let X be a strong KS-semigroup then :

1- $x.y*y=0 \quad for\ all \quad x, y \in X$.

2- $x*y=0 \leftrightarrow x*x.y=0 \quad for\ all \quad x, y \in X$.

Definition (2. 7) [11] A non empty subset I of a BCK –algebra X is called a ***H- ideal*** of X if the following conditions hold :

1- $0 \in I$.

2- If $x*(y*z) \in I \quad and \quad y \in I \Rightarrow x*z \in I$, *for all* $x, y, z \in X$

Definition(2. 8) [7] Let X and Y be KS-semigroups . a mapping $f : X \to Y$ is called a ***KS-Semigroup***. ***homomorphism*** (briefly ***homomorphism***) if $f(x*y) = f(x)*f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in X$. Let $f : X \to Y$ KS-Semigroup homomorphism . then the set $\{x \in X / f(x) = 0\}$ is called ***the kernel of f*** , and denote by ***ker f*** . moreover the set $\{f(x) \in Y / x \in X\}$ is called ***the image of f*** and denote by ***Im f*** .

Definition (2. 9) [9] A non-empty subset A of a semigroup (X, .) is said to be ***left (resp. right) stable*** *if* $xa \in A$ *(resp.*$ax \in A$*) whenever* $x \in X$ and $a \in A$.

Both left and right stable is called ***two-sided stable*** or simply *stable*.

Definition. (2.10) [9]A non-empty subset A of a KS-semigroup X is said to be ***left (resp. right) ideal*** of X if :

1. A is left (resp.right) stable subset of (X,.) and

2. $x * y \in A$ *and* $y \in A$ *imply that* $x \in A$, *for all* $x, y \in X$.

If A is both left and right ideal then A is called two-sided ideal or simply an ideal .

Remarks (2.11)

■ *let* A be a KS-ideal then $0 \in A \quad for\ all \ x \in X$ since $A \neq \theta$ *then* $\exists\ a \in A$ *such that* $xa , ax \in A$, *put* $x = 0$ *we get* $0 \in A$

■ *let* $f : X \to Y$ KS-Semigroup homomorphism then $f(0) = 0$ and if $x \le y$, then $f(x) \le f(y)$,[7] .

■ ker f is a KS-ideal[7] .

Definition (2.12) [9] A non-empty subset A of a KS-Semigroup X is said to be ***left (resp.right) p-ideal*** of X if :

1. A is a left (resp. right) stable subset of (X, .) and,

2. $(x * y) * z \in A$ *and* $y * z \in A$ *imply that* $x * z \in A$, *for all* $x, y, z \in X$.

If A is both left and right p- ideal then A is called *two sided ideal* or *simply p-ideal*

Theorem (2.13) [7] Every p-ideal of a *KS-Semigroup* X is an ideal but convers is not true

Definition (2.14) [10] The element e is called a unity in a KS-semigroup X if $e.x = x.e = x \quad \forall x \in X$.

Definition (2.15) [1] Let X be a non-empty set a ***fuzzy subset*** of X is a function $\mu : X \to [0, 1]$.

Remarks (2.16)[1]

Let X be a non-empty set then :

1) each fuzzy subset λ and μ of X , if $\lambda \subseteq \mu$ mean that $\lambda(a) \le \mu(a) \quad for\ all \ a \in X$.

2) *if* $x \le y$ *implies that* $\mu(x) \ge \mu(y) \quad for\ all \quad x, y \in X$.

3) If μ, ν be two fuzzy set of X and $a \le b$ such that $a, b \in [0,1]$, then $\mu_b \subseteq \mu_a$.

Definition (2.17) [9] Let X be a non-empty set and let μ be the fuzzy subset of X for a fixed $0 \le t \le 1$, the set $\mu_t = \{x \in X \ / \ \mu(x) \ge t\}$ is called an ***upper level set*** of μ

Definition (2. 18) [9] Let $f : X \to Y$ be a mapping of KS-Semigroup and μ be a fuzzy subset of Y . The map μ^f is ***the pre-image of*** μ under f if $\mu^f = \mu(f(x)) \ \forall x \in X$.

Definition (2.19) [5] Let X be a BCK –algebra a fuzzy subset μ of X is called a ***fuzzy subalgebra*** of X if it satisfies the fallowing condition : $\mu(x * y) \ge \min\{\mu(x), \mu(y)\} \quad \forall x, y \in X$.

Definition (2.20) [11] A fuzzy set μ of BCK –algebra X is called a ***fuzzy H-ideal*** if it satisfies :

1- $\mu(0) \ge \mu(x) \quad \forall x \in X$,

2- $\mu(x * z) \ge \min\{\mu(x * (y * z)), \mu(y)\} \quad \forall \ x, y, z \in X$.

Definition (2.21) [9] A fuzzy set μ defined on X is called a ***fuzzy subKS-semigroup*** of X if it satisfies the fallowing conditions :

1. $\mu(x_1 * x_2) \ge \min\{\mu(x_1), \mu(x_2)\}$,
2. $\mu(x_1 x_2) \ge \min\{\mu(x_1), \mu(x_2)\} \quad \forall \ x_1, x_2 \in X$

Definition (2.22) [9] A fuzzy subset μ of X is called a ***left fuzzy KS-ideal*** if :

KSI1. $\mu(0) \ge \mu(x)$

KSI2. $\mu(x) \ge min\{\mu(x * y), \mu(y)\}$

KSI3. $\mu(xa) \ge min\{\mu(x), \mu(a)\}$ *for all* $x, y, a \in X$.

A fuzzy subset μ is called a ***right fuzzy KS-ideal*** if it satisfies **KSI1, KSI2** and **KSI4:** $\mu(ax) \ge min\{\mu(x), \mu(a)\}$, for all $x, y, a \in X$.

A fuzzy subset μ of X is called a ***fuzzy KS-ideal*** if it is both left and right fuzzy KS-ideal of X.

Definition (2.23) [9] A fuzzy subset μ of X is called a ***left fuzzy p-ideal*** if :

KSP1. $\mu(0) \ge \mu(x)$

KSP2. $\mu(x * z) \ge min\{\mu((x * y) * z), \mu(y * z)\}$

KSP3. $\mu(xa) \ge min\{\mu(x), \mu(a)\}$ for all $x, y, z, a \in X$.

A fuzzy subset μ is called a ***right fuzzy p-ideal*** if it satisfies **KSP1, KSP2** and **KSP4 :** $\mu(ax) \ge min\{\mu(x), \mu(a)\}$ for all $x, y, a \in X$.

A fuzzy subset μ of X is called a ***fuzzy p-ideal*** if it is both left and right fuzzy p-ideal of X.

Theorem (2.24) [9] Every left (resp.right) fuzzy p-ideal of X is a left (resp.right) fuzzy KS-ideal of X .

Definition (2.25) [9] Let λ and μ be the fuzzy subsets in a set X The ***cartesian product*** $\lambda \times \mu : X \times X \longrightarrow [0, 1]$ is defined by $(\lambda \times \mu)(x, y) = min\{\lambda(x), \mu(y)\}$ for all $x, y \in X$.

Definition (2.26) [9] Let ν be a fuzzy subset in X ***the strong fuzzy relation*** on X that is a fuzzy relation on ν is ρ_ν given by $\rho_\nu(x, y) = \min\{\nu(x), \nu(y)\}$

3. *KS-H-Ideal*

Definition (3.1)

A non-empty subset I of a KS-semigroups X is said to be ***left KS-H-ideal*** of X if it satisfies :

1) *If* $x * (y * z) \in I$ *and* $y \in I$ *then* $x * z \in I$

2) $xa \in I$ (*resp.* $ax \in I$) *whenever* $x \in X$ *and* $a \in I$.

A non-empty subset I is said to be ***right KS-H-ideal*** of X if it satisfies (1) and (3) : $ax \in I$ *whenever* $x \in X$ *and* $a \in I$.

If I is both left and right KS-H- ideal then I is called *two-sided KS-H- ideal* or simply *KS-H- ideal* .

Example(3. 2)

Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4, No.14, 2014

Let X = {0, 1, 2, 3} be defined by the following tables:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then by usual calculations we can prove that X *is a* ***KS-semigroup***. If A ={0,1} then A is a *KS-H- ideal* of a KS-semigroup X .

Proposition (3. 3)

Let X be a KS-semigroup and let A be left (resp. right) KS-H-ideal of X then A is a left (resp. right) KS-ideal of X .

Proof:

Let A be a left KS-H-ideal of X then A is a stable. Now, let $x, y \in X$ such that $x*y \in A$ *and* $y \in A$ *then* $x*y = x*(y*0) \in A$ *and* $y \in A$ *then* $x \in A$ and since A is a left KS-H-ideal then .Hence A is a left KS-H-ideal

.

Proposition (3.4)

Let I and J are left (resp. right) KS-H-ideal of KS-Semigroups X then $I \cap J$ is a left (resp. right) KS-H-ideal of X .

Proof: *it is clear*

Proposition (3.5)

Let I and J are left (resp. right) KS-H-ideal of KS-Semigroups X then $I \cup J$ is a left (resp. right) KS-H-ideal *if* $I \subseteq J$ *or* $J \subseteq I$.

Proof: it is clear

Proposition (3. 6)

Let I and J are left (resp. right) KS-H-ideal of KS-Semigroups X then $I \times J$ is a left (resp. right) KS-H-ideal of $X \times X$.

Proof:

Let I and J are left KS-H-ideal of KS-Semigroups X

For any $x_1, x_2, a_1, a_2 \in X$ and $(x_1, x_2) \in X \times X$, $(a_1, a_2) \in I \times J$ then

$(x_1, x_2).(a_1, a_2) = (x_1 a_1, x_2 a_2)$, since I, J *are left* $KS-H-ideal$ so $x_1 a_1 \in I$ *and* $x_2 a_2 \in J$

then $(xa_1, xa_2) \in I \times J$ therefore $(x_1, x_2).(a_1, a_2) \in I \times J$

let $x*(y*z) \in I \times J$ *and* $y \in I \times J$ *, where* $x = (x_1, x_2)$, $y = (y_1, y_2)$ *and* $z = (z_1, z_2) \in X \times X$

if $(x_1, x_2)*[(y_1, y_2)*(z_1, z_2)] \in I \times J$ *and* $(y_1, y_2) \in I \times J$ *then* $(x_1, x_2)*(y_1*z_1, y_2*z_2) \in I \times J$ *and* $(y_1, y_2) \in I \times J$

then $(x_1*(y_1*z_1), x_2*(y_2*z_2) \in I \times J$ *and* $(y_1, y_2) \in I \times J$ *then* $(x_1*(y_1*z_1)) \in I$, $(x_2*(y_2*z_2)) \in J$

, $y_1 \in I$ *and* $y_2 \in J$ *then* $x_1*z_1 \in I$ *and* $x_2*z_2 \in J$ [since I, J *are left* $KS-H-ideal$]so $(x_1*z_1, x_2*z_2) \in I \times J$

so $(x_1, x_2)*(z_1, z_2) \in I \times J$ *then* $x*z \in I \times J$

hence $I \times J$ is a left KS-H-ideal .

Proposition (3. 7)

Let $f: \mathrm{X} \to \mathrm{Y}$ be a KS-semigroup epimorphism if A is a left (resp. right) KS-H-ideal in X then $f(A)$ is a left (resp. right) KS-H-ideal in Y .

Proof:

Let A be a left KS-H-ideal of X . let $a^{-} = f(a) \in f(A)$ *and* $y \in Y$ *where* $a \in A$
Since f onto then there exists $x \in X$ *such that* $f(x) = y$
since $xa \in A$ $\forall\ x \in \mathrm{X}$ *and* $a \in A$ so $f(xa) \in f(A)$ but $f(xa) = f(x)f(a) = ya^{-}$ [since f is epimorphism]

therefore $f(A)$ is stable . Now , Suppose that $f(x), f(y), f(z) \in f(A)$ *for some* $x, y, z \in A$ Such that $f(x)*[f(y)*f(z)] \in f(A)$ *and* $f(y) \in f(A)$, since f is a homomorphism then $f(x)*[f(y)*f(z)] = f(x*(y*z)) \in f(A)$ *and* since $f(y) \in f(A)$,
thus $x*(y*z) \in A$, $y \in A \to x*z \in A$ [since A is KS-H-ideal] therefore $f(x*z) \in f(A)$ but $f(x)*f(z) = f(x*z) \in f(A)$ so hence $f(A)$ is a left KS-H-ideal .

<u>Proposition (3. 8)</u>

Let $f: \mathrm{X} \to \mathrm{Y}$ be a KS-semigroup homomorphism then ker f is a KS-H-ideal of X .

Proof:

Let $f: X \to Y$ be a KS-semigroup homomorphism , since ker f is an ideal of X [3] it follows that ker f is a stable , now, let $x, y, z \in X$ such that $x*(y*z) \in \ker f$ *and* $y \in \ker f$,
so $f(x*(y*z)) = 0$ *and* $f(y) = 0$ so $f(x)*[f(y)*f(z)] = 0$ *and* $f(y) = 0$ so $f(x)*[0*f(z)] = 0$
so $f(x) = 0$ so $x \in$ ker f ,now , $f(x*z) = f(x)*f(z) = 0*f(z) = 0$
therefore $x*z \in \ker f$ hence ker f is a KS-H-ideal .

<u>Proposition (2.1.9)</u>

Let I be a KS-ideal of KS-semigroup X such that $x*y = y*x$ for all $x \neq 0$ *and* $y \neq 0$ and $x*y = 0$ just when $x = 0$. Then I is a KS-H-ideal of X .

Proof:

First since I is a KS-ideal so $xa \in I$ $\forall x \in \mathrm{X}$ *and* $a \in I$, Now
let $x, y, z \in \mathrm{X}$ *and* $x*(y*z), y \in I$ *to prove* $x*z \in I$. There are several cases :

1) *If* $x, y, z \neq 0$ *and* $x \neq y \neq z$ *so*

$$
\begin{aligned}
x*(y*z) &= x*(z*y) && [\text{since } x*y = y*x\ \forall\ x, y \neq 0] \\
&= (z*y)*x && [\text{since } x*y = y*x\ \forall\ x, y \neq 0 \text{ and } x*y \neq 0] \\
&= (z*x)*y && [\text{since } (x*y)*z = (x*z)*y \text{ in } BCK] \\
&= (x*z)*y \in I && [\text{since } x*y = y*x\ \forall\ x, y \neq 0 \text{ and } x*y \neq 0\ ]
\end{aligned}
$$

and $y \in I$ *then* $x*z \in I$ [*since* I *is a* $KS-ideal$].

2) *If* $x = 0$ *and* $y, z \neq 0$ *so*
$0*(y*z) \in I$ *and* $y \in I$ *then*
$0 \in I$ *and* $y \in I$, I *is a* $KS-ideal$ *so*
$x*z = 0*z \in I$.

3) *If* $x = y = 0$ *and* $z \neq 0$ *then*
$0*(0*z) \in I$ *and* $z \in I$ *so* $x*z = 0*z \in I$.

4) *If* $x = 0, z = 0$ *then* $x*z \in I$ [*by the same way of* (3)].

5) $If\ x \neq 0$, $and\ y = 0\ then$
$x*(0*z) \in I$,$0 \in I$ $then\ x \in I, 0 \in I$.

$so\ If\ z = 0\ then\ x*z = x \in I\ and$
$If\ z \neq 0\ then\ (x*z)*x = 0\ and\ x \in I\ so\ x*z \in I\ [I\ is\ a\ KS-ideal]$.

6) $If\ x = 0, y = 0, z = 0\ then\ 0*(0*0) = 0 \in I$ so $x*z \in I$.

7) $If\ x \neq 0$, $y \neq 0$, $z = 0\ so$
$x*(y*0) \in I\ and\ y \in I\ then\ x*y \in I\ and\ y \in I\ [since\ I\ is\ a\ KS-ideal]$
$so\ x*z = x*0 \in I$

8) $If\ x = 0, y \neq 0, z = 0\ then\ 0*(y*0) \in I\ so\ x*z \in I$.

9) $If\ x = 0,, z \neq 0, y \neq 0\ then\ 0*(y*z) \in I\ so\ x*z \in I$.

10) $If\ z = 0, x \neq 0, y = 0\ \ x*(y*z) = x*0 \in I\ and\ z = 0 \in I \rightarrow x*z = x*0 \in I$, [since I is a KS-ideal] .

Then I is a KS-H-ideal .

4. ***fuzzyKS-H-Ideal***
In this section , we define the notion of the fuzzy KS-H-ideal of KS-semigroup X and and prove some results and examples.

Definition (4..1)
A fuzzy subset μ of KS-semigroup X is called a ***left fuzzy KS-H-ideal*** if the following conditions hold :
$KSH1\ \ \mu(0) \geq \mu(x)$,
$KSH2\ \ \mu(x*z) \geq \min\{\mu(x*(y*z)), \mu(y)\}$,
$KSH3\ \ \mu(xa) \geq \min\{\mu(x), \mu(a)\}$, $for\ all\ x, y, z, a \in X$.
A fuzzy subset μ is called a ***right fuzzy KS-H-ideal*** if it satisfies $KSH1$, $KSH2$ and
$KSH4$: $\mu(ax) \geq \min\{\mu(x), \mu(a)\}$, $for\ all\ x, y, a \in X$.
A fuzzy subset μ is called a ***fuzzy KS-H-ideal*** it if is both left and right fuzzy KS -H-ideal of X .
Example (4.2)

Let X = {0, 1, 2, 3} be a KS-semigroup defined by the following tables:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	3	2	0	0
3	3	3	3	0

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Define a fuzzy subset $\mu : X \rightarrow [0,1]$ by $\mu(0) = 0.4$, $\mu(x) = 0.2\ \ \forall x \neq 0 \in X$. by usual calculations, we can prove that μ is a left fuzzy KS-H-ideal of X .
Remark(4.3)
Every fuzzy KS-H-ideal is a fuzzy KS- ideal .

Proof:
let μ be a fuzzy KS-H-ideal of X since $x*0 = x\ \ \forall x \in X$ [by Remark (2.2)]

$\mu(x) = \mu(x*0) \geq \min\{\mu(x*(y*0)), \mu(y)\}$
$= \min\{\mu(x*y), \mu(y)\}$

thus $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$

and since $\mu(0) \geq \mu(x)\ \ \forall x \in X$ and $\mu(xa) \geq \min\{\mu(x), \mu(a)\}, \mu(ax) \geq \min\{\mu(a), \mu(x)\}\ \ \forall x, a \in X$

Hence μ is a fuzzy KS- ideal .

Proposition (4.4)

Let μ and λ are left (resp. right) fuzzy KS-H-ideal of KS-semigroup X then $\mu \cap \lambda$ is a left (resp. right) fuzzy KS-H-ideal .

Proof: Let μ and λ are left fuzzy KS-H-ideal of X then

$(\mu \cap \lambda)(0) = \min\{\mu(0), \lambda(0)\} \geq \min\{\mu(x), \lambda(x)\} = (\mu \cap \lambda)(x)\ \ \forall x \in X$ [$\sin ce\ \mu, \lambda\ are\ left\ \ fuzzy\ KS-H-ideal$]

$, now, (\mu \cap \lambda)(xa) = \min\{\mu(xa), \lambda(xa)\} \geq \min\{\min\{\mu(x), \mu(a)\}, \min\{\lambda(x), \lambda(a)\}\} = \min\{\min\{\mu(x), \lambda(x), \min\{\mu(a), \lambda(a)\}$
$= \min\{(\mu \cap \lambda)(x), (\mu \cap \lambda)(a)\}\ \ \forall x, a \in X$

$so, (\mu \cap \lambda)(x*z) = \min\{\mu(x*z), \lambda(x*z)\} \geq \min\{\min\{\mu(x*(y*z), \mu(y)\}, \min\{\lambda(x*(y*z), \lambda(y)\}\}$
$= \min\{\min\{\mu(x*(y*z), \lambda(x*(y*z)\}, \min\{\mu(y), \lambda(y)\}\}$
$= \min\{\mu(x*(y*z)), \lambda(x*(y*z))\}, \min\{\mu(y), \lambda(y)\}\}\ \ \forall\ x,y,z \in X$

hence $\mu \cap \lambda$ is a left fuzzy KS-H-ideal .

Proposition(4.5)

Let μ and ν be two fuzzy KS-H-ideal of KS-semigroup X if $\mu \subseteq \nu$ *or* $\nu \subseteq \mu$ then $\mu \cup \nu$ is a fuzzy KS-H-ideal .

Proof:

Let μ and ν are fuzzy KS-H-ideal of X , without loss of generality we may assume that let $\mu \subseteq \nu$
since μ and ν are fuzzy KS-H-ideal and $x, y, a \in X$ so $\mu(0) \geq \mu(x)$ *and* $\nu(0) \geq \nu(x)$, $\forall\ x \in X$ therefore,
$(\mu \cup \nu)(0) = \max\{\mu(0), \nu(0)\} \geq \max\{\mu(x), \nu(x)\} = (\mu \cup \nu)(x)$,now ,since μ and ν are fuzzy KS-H-ideal so
$\mu(xa) \geq \min\{\mu(x), \mu(a)\}$ and $\nu(xa) \geq \min\{\nu(x), \nu(a)\}$
$\max\{\mu(xa), \nu(xa)\} \geq \max\{\min\{\mu(x), \mu(a)\}, \min\{\nu(x), \nu(a)\}\}$ since $\mu \subseteq \nu$ therefore
$(\mu \cup \nu)(xa) \geq \min\{\max\{\mu(x), \mu(a)\}, \max\{\nu(x), \nu(a)\}\} = \min\{\max\{\mu(x), \nu(x)\}, \max\{\mu(a), \nu(a)\}\} = \min\{(\mu \cup \nu)(x), (\mu \cup \nu)(a)\}$
and so, since $\mu(x*z) \geq \min\{\mu(x*(y*z)), \mu(y)\}$ and $\nu(x*z) \geq \min\{\nu(x*(y*z)), \nu(y)\}$ so
$\max\{\mu(x*z), \nu(x*z)\} \geq \max\{\min\{\mu(x*(y*z)), \mu(y)\}, \min\{\nu(x*(y*z)), \nu(y)\}\}$ since $\mu \subseteq \nu$ therefore
$(\mu \cup \nu)(x*z) \geq \min\{\max\{\mu(x*(y*z)), \mu(y)\}, \max\{\nu(x*(y*z)), \nu(y)\}\}$
$= \min\{\max\{\mu(x*(y*z)), \nu(x*(y*z))\}, \max\{\mu(y), \nu(y)\}\} = \min\{(\mu \cup \nu)(x*(y*z)), (\mu \cup \nu)(y)\}$

hence $\mu \cup \nu$ is a fuzzy KS-H-ideal .

Proposition (4.6)

Let I and J are left (resp. right) fuzzy KS-H-ideal of KS-semigroup X then $I \times J$ is a left (resp. right) fuzzy KS-H-ideal of $X \times X$.

Proof:

Let I and J are left fuzzy KS-H-ideal of X then
$(I \times J)(0,0) = \min\{I(0), J(0)\} \geq \min\{I(x), J(y)\} = (I \times J)(x, y)\ \ \forall (x, y) \in X \times X.$ [since I,J are left fuzzy KS-H-ideal of X], let $(x, x) \in X \times X$ *and* $(a_1, a_2) \in I \times J$ so ,
$(I \times J)(x, x)(a_1, a_2) = (I \times J)(xa_1, xa_2) = \min\{I(xa_1), J(xa_2)\} \geq \min\{\min\{I(x), I(a_1)\}, \min\{J(x), J(a_2)\}\}$
$= \min\{\min\{I(x), J(x)\}, \min\{I(a_1), J(a_2)\}\} = \min\{(I \times J)(x, x), (I \times J)(a_1, a_2)\}$

$now,\ let\ \ (x_1, x_2)\ , (y_1, y_2)\ \ and\ \ (z_1, z_2) \in X \times X,$
$(I \times J)((x_1, x_2)*(z_1, z_2)) = (I \times J)(x_1 * z_1, x_2 * z_2) = \min\{I(x_1 * z_1), J(x_2 * z_2)\}$
$\geq \min\{\min\{I(x_1 * (y_1 * z_1), I(y_1)\}, \min\{J(x_2 * (y_2 * z_2)), J(y_2)\}\}$
$= \min\{\min\{I(x_1 * (y_1 * z_1), J(x_2 * (y_2 * z_2)\}, \min\{I(y_1), J(y_2)\}\}$
$= \min\{(I \times J)((x_1, x_2)*(y_1 * z_1, y_2 * z_2)), (I \times J)(y_1, y_2)\}$
$= \min\{(I \times J)((x_1, x_2)*((y_1, y_2)*(z_1, z_2))), (I \times J)(y_1, y_2)\}$

hence $I \times J$ is a left fuzzy KS-H-ideal .

Proposition (4.7)

If A be a left (resp. right) KS-H-ideal of KS-semigroup X then $\forall\ 0 < t \leq 1$ their exist a left (resp. right) fuzzy KS-H-ideal μ such that $A = \mu_t$.

Proof:

Let A be a left KS-H-ideal and μ be defined by

$$\mu(x) = \begin{cases} t & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases} \qquad where \quad o < t \leq 1$$

$let\ x \in A \quad then\ \ \mu(x) = \mathrm{t}\ , then \quad x \in \mu_t\ , so \quad A \subseteq \mu_t\ , and\ \ if \quad x \in \mu_t \quad then \quad \mu(x) \geq t\ , then \quad x \in A \qquad so\ \ A = \mu_t$

Since A is a left KS-H-ideal $so\ 0 \in A then\ \ \mu(0) = t \geq \mu(x)\ \ \forall x \in \mathrm{X}$, $now\ let\ \ x, a \in \mathrm{X}$ there are several cases :

1. $If\ \ x, a \in X\ so$ so $xa \in A\ since\ A\ is\ a\ left\ H - ideal \quad \mu(xa) = t \geq \min\{\mu(x), \mu(a)\}$. so ,
2. $If\ x \notin A\ \ and \quad a \notin A \qquad then \quad \mu(xa) \geq \min\{\mu(x), \mu(a)\} = 0$
3. $If\ \ at\ most\ one\ of\ \ x, a\ \ belong\ to\ A\ , then\ \ at\ most\ one\ of\ \mu(x)\ and\ \mu(a)$ $is\ equal\ \ to\ 0\ .\ therefore \quad \mu(xa) \geq \min\{\mu(x), \mu(a)\} = 0$

$\mu(xa) \geq \min\{\mu(x), \mu(a)\} \quad \forall\ x, a \in \mathrm{X}\ let \quad x*(y*z)\ , y \in \mathrm{X}\ there\ are\ saveral\ cases:$

1. $If\ \ x*(y*z), y \in A then\ \ x*z \in A since\ \ A\ is\ a\ left\ \ KS - H - ideal\ \ so\ \ (x*z) = t \geq \min\{\mu(x*(y*z)), \mu(y)\}$
2. $If\ \ x*(y*z), y \notin A \quad then \quad \mu(x*(y*z)) = \mu(y) = 0 \quad so \quad \mu(x*z) \geq \min\{\mu(x*(y*z)), \mu(y)\} = 0$
3. $If\ \ at\ most\ one\ of\ \ x*(y*z), y\ belong\ to\ A, then\ at\ most\ one\ of\ \mu(x*(y*z)) \quad and \quad \mu(y)\ \ is\ equal\ to\ 0, therefor$ $x*z \notin A \quad \mu(x*z) \geq \min\{\mu(x*(y*z)), \mu(y)\} = 0\ \ so\ \ \mu(x*z) \geq \min\{\mu(x*(y*z)), \mu(y)\} \qquad for\ all\ \ x, y, z \in \mathrm{X}.$

hence μ is a left fuzzy KS-H-ideal .

Proposition (4.8)

Let μ be a left (resp. right) fuzzy KS-H-ideal in KS-semigroup X then a fuzzy set μ^+ defined by $\mu^+ = \mu(x) + 1 - \mu(0)$ is a left (resp. right) fuzzy KS-H-ideal such that $\mu \subseteq \mu^+$.

Proof:

Let μ be a left fuzzy KS-H-ideal and μ^+ is a fuzzy set then

$\mu^+(0) = \mu(0) + 1 - \mu(0) = 1 \geq \mu^+(x) \quad \forall\ x \in \mathrm{X}$. now , let $x, a \in \mathrm{X}$ so

$$\begin{aligned} \mu^+(xa) &= \mu(xa) + 1 - \mu(0) \\ &\geq \min\{\mu(x), \mu(a)\} + 1 - \mu(0) \quad [since\ \ \mu\ \ is\ \ a\ \ left\ \ fuzzy\, KS - H - ideal] \\ &= \min\{\mu(x) + 1 - \mu(0), \mu(a) + 1 - \mu(0)\} = \min\{\mu^+(x), \mu^+(a)\}. \end{aligned}$$

let $x, y, z \in \mathrm{X}$ so

$$\begin{aligned} \mu^+(x*z) &= \mu(x*z) + 1 - \mu(0) \\ &\geq \min\{\mu(x*(y*z)), \mu(y)\} + 1 - \mu(0) \quad [since\ \ \mu\ \ is\ \ a\ \ left\ \ fuzzy\, KS - H - ideal] \\ &= \min\{\mu(x*(y*z)) + 1 - \mu(0), \mu(y) + 1 - \mu(0)\} \\ &= \min\{\mu^+(x*(y*z)), \mu^+(y)\} \end{aligned}$$

hence μ^+ is a left fuzzy KS-H-ideal .

Proposition (4.9)

Let $f : X \to Y$ be a homomorphism if μ is a left (resp. right) fuzzy KS-H-ideal of Y then μ^f is a left (resp. right) fuzzy KS-H-ideal of X .

Proof :

Let μ be a left fuzzy KS-H-ideal of Y then $\mu^f(0) = \mu(f(0)) \geq \mu(f(x)) = \mu^f(x) \quad \forall\ x \in X$.

now, let $x, a \in X$ so

$$\mu^f(xa) = \mu(f(xa)) = \mu(f(x)f(a)) \qquad [f \text{ is a homomorphism}]$$
$$\geq \min\{\mu(f(x)), \mu(f(a))\} \qquad [\text{since } \mu \text{ is a left fuzzy } KS-H-ideal]$$
$$= \min\{\mu^f(x), \mu^f(a)\}$$

let $x,y,z \in X$, $\mu^f(x*z) = \mu(f(x*z)) = \mu(f(x)*f(z))$

$$\geq \min\{\mu(f(x)*[f(y)*f(z)]), \mu(f(y))\} \qquad [\text{since } \mu \text{ is a left } H-ideal]$$
$$= \min\{\mu(f(x*(y*z)), \mu(f(y))\}$$
$$= \min\{\mu^f(x*(y*z)), \mu^f(y)\}$$

hence μ^f is a left fuzzy KS-H-ideal of X .

Proposition (4.10)

Let $f : X \to Y$ be an epimorphism if μ^f is a left (resp. right) fuzzy KS-H-ideal of X then μ is a left (resp. right) fuzzy KS-H-ideal of Y.

Proof :

Let μ^f is a fuzzy KS-H-ideal of X and

let $y \in Y$ *then* $\exists\ x \in X$ *such that* $f(x) = y$

$$\mu(y) = \mu(f(x)) = \mu^f(x) \leq \mu^f(0) \qquad [\text{since } \mu^f \text{ is a left fuzzy } KS-H-ideal]$$
$$= \mu(f(0)) = \mu(0) . \quad [by \text{ remark 2.13}]$$

now, let $x, a \in Y$ *then* $\exists\ t, m \in X$ *such that*

$f(t) = x$, $f(m) = a$ *then* $\mu(xa) = \mu(f(t)f(m)) = \mu(f(tm))$

$$= \mu^f(tm) \geq \min\{\mu^f(t), \mu^f(m)\} = \min\{\mu(f(t)), \mu(f(m))\} = \min\{\mu(x), \mu(a)\}$$

so, let $x, y, z \in Y$ *then* $\exists\ a, b, c \in X$ *such that* $f(a) = x$, $f(b) = y$, $f(c) = z$ *then*

$$\mu(x*z) = \mu(f(a)*f(c)) = \mu(f(a*c)) = \mu^f(a*c)$$
$$\geq \min\{\mu^f(a*(b*c)), \mu^f(b)\} = \min\{\mu(f(a*(b*c)), \mu(f(b))\} = \min\{\mu(f(a)*[f(b)*f(c)]), \mu(f(b)\}$$
$$= \min\{\mu(x*(y*z)), \mu(y)\}$$

hence μ is a fuzzy KS-H-ideal of Y.

Proposition (4.13)

Let I be a non-empty subset of a strong KS-semigroup X then I is a left (resp. right) KS-H-ideal of X if and only if χ_I is a left (resp. right) fuzzy KS-H-ideal where $\chi_I : X \to [0,1]$ define as follows :

$$\chi_I = \begin{cases} 1 & if \quad x \in I \\ 0 & if \quad x \notin I \end{cases}$$

Proof:

It is clear that χ_I is a fuzzy set .

suppose that I is a left KS-H-ideal of X and $x, y, a \in X$

since $0 \in X$ *so* $0.a = 0 \in I \quad \forall\, a \in I$ *then* $\chi_I(0) = 1 \geq \chi_I(x) \quad \forall\ x \in X.$

there are several cases : *let* $x, a \in X$

$1-If\ x\in I, a\in I\quad so\ xa\in I\ \ [since\ I\ is\ a\ left\ fuzzy\ KS-H-ideal]$
$\chi_I(x)=1\ ,\chi_I(a)=1\ \ and\ \ \chi_I(xa)=1\ \ then\ \ \chi_I(xa)\geq\min\{\chi_I(x),\chi_I(a)\}$

$2-If\ x\in I, a\notin I\quad so\ xa\notin I\quad thus\ \chi_I(x)=1\ ,\chi_I(a)=0\ \ and\ \ \chi_I(xa)=0$
$then\ \ \chi_I(xa)\geq\min\{\chi_I(x),\chi_I(a)\}$
$3-If\ x\notin I, a\in I\quad so\ xa\in I\quad thus\ \chi_I(x)=0\ ,\chi_I(a)=1\ \ and\ \ \chi_I(xa)=1$
$then\ \ \chi_I(xa)\geq\min\{\chi_I(x),\chi_I(a)\}$

$4-If\ x\notin I, a\notin I\quad so\ xa\notin I\quad thus\ \chi_I(x)=0\ ,\chi_I(a)=0\ \ and\ \ \chi_I(xa)=0$
$then\ \ \chi_I(xa)\geq\min\{\chi_I(x),\chi_I(a)\}$

$so\ \ \chi_I(xa)\geq\min\{\chi_I(x),\chi_I(a)\}$

In similar way we can prove that $\chi_I(x*z)\geq\min\{\chi_I(x*(y*z)),\chi_I(y)\}\quad\forall\ x,y,z\in X.$

Hence χ_I is a left KS-H-ideal .

Conversely, assume that X is a strong KS-semigroup and let χ_I is a fuzzy KS-H-ideal of X

$let\ x\in X\quad and\quad a\in I\ since\ X\ is\ a\ strong\ KS-semigroup\ so\quad xa*a=0\quad and\ since\ \ 0\in I\quad so$
$xa*a\in I\quad and\quad a\in I\quad then\quad \chi_I(xa)\geq\min\{\chi_I((xa*a),\chi_I(a)\}=\min\{\chi_I(0),\chi_I(a)\}=\chi_I(a)=1$
$so\quad xa\in I$
$now,\ let\quad x*(y*z)\in I\quad and\quad y\in I\ \ so$
$\chi_I(x*(y*z))=\chi_I(y)=1\quad since\quad \chi_I\ \ is\ fuzzy\ KS-H-ideal\ we\ have$
$\chi_I(x*z)\geq\min\{\chi_I(x*(y*z)),\chi_I(y)\}=1\ \ so\ \ x*z\in I$

therefore *I* is a left KS-H-ideal

Proposition (4.14)

If μ be a right fuzzy KS-H-ideal of KS-semigroup X with left identity *e* and satisfying the condition $(xy)z=(xz)y\quad\forall\ x,y,z\in X$ then μ is a left fuzzy H- ideal of X.

Proof:

Let μ be a right fuzzy KS-H-ideal of KS-semigroup X with left identity and let $x,a\in X$

$\mu(xa)=\mu((ex)a)=\mu((ea)x)=\mu(ax)\geq\min\{\mu(a),\mu(x)\}\qquad[by\ hypothesis]$
$\mu(xa)\geq\min\{\mu(x),\mu(a)\}$
$since\quad \mu(0)\geq\mu(x)\quad\forall x\in X\quad and$
$\mu(x*z)\geq\min\{\mu(x*(y*z)),\mu(y)\}\qquad[\mu\ is\ a\ right\ fuzzy\ KS-H\ ideal]$

therefore μ is a left fuzzy KS- H- ideal.

Corollary (4.15)

Every right fuzzy KS-H-ideal of KS-semigroup X with left identity *e* satisfying the condition is a fuzzy KS-H- ideal of X .

Proof:

Let μ be a right fuzzy KS-H- ideal with left identity then μ is a left fuzzy KS -H- ideal therefore μ is a fuzzy KS-H-ideal .

Proposition (4.16)

Let μ be a fuzzy set of strong KS-semigroup X if μ is a left fuzzy KS-H-ideal then μ_t left KS-H-ideal where $t\in[0,\mu(0)]$.

Proof:

Let μ be a left fuzzy KS-H-ideal of X ,and $t\in[0,\mu(0)]$. let $x\in\mu_t$

since $\mu(0) \geq t$ *then* $0 \in \mu_t$ then $\mu_t \neq \varphi$,
now, let $x \in X$ *and* $a \in \mu_t$ *so* $\mu(a) \geq t$ since X is a strong so $xa * a = 0$ and
since μ is a left fuzzy KS-H-ideal so
$\mu(xa) \geq \min\{\mu(xa*(a*0)), \mu(a)\} = \min\{\mu(xa*a), \mu(a)\} = \min\{\mu(0), \mu(a)\} = \mu(a) \geq t$
$\Rightarrow$ $xa \in \mu_t$
let $x*(y*z) \in \mu_t$ *and* $y \in \mu_t$ then $\mu(x*(y*z)) \geq t$ *and* $\mu(y) \geq t$, since μ is a left fuzzy KS-H-ideal so
$\mu(x*z) \geq t$ *so* $x*z \in \mu_t$ .Therefore μ_t is a left KS-H-ideal .

Theorem (4.17)

Let X be a KS-semigroup and μ , λ be two fuzzy sets in X such that $\mu \times \nu$ is a fuzzy KS-H-ideal of X then :

1. either $\mu(x) \leq \mu(0)$ *or* $\lambda(x) \leq \lambda(0)$ *for all* $x \in X$.
2. If $\mu(x) \leq \mu(0)$ *for all* $x \in X$ then either $\mu(x) \leq \lambda(0)$ *or* $\lambda(x) \leq \lambda(0)$.
3. If $\lambda(x) \leq \lambda(0)$ *for all* $x \in X$ then either $\mu(x) \leq \mu(0)$ *or* $\lambda(x) \leq \mu(0)$.
4. either μ or λ is a fuzzy KS-H-ideal of X .

Proof:

since $\mu \times \nu$ is a fuzzy KS-H-ideal of X then it is fuzzy sub KS semigroup by [ ], so (1),(2) and (3) satisfied by [12]. Now, to prove 4, Since by (1) either $\mu(x) \leq \mu(0)$ *or* $\lambda(x) \leq \lambda(0)$ *for all* $x \in X$ without loss of generality we may assume that $\lambda(x) \leq \lambda(0)$ it follows from (3) that either $\mu(x) \leq \mu(0)$ *or* $\lambda(x) \leq \mu(0)$ if $\lambda(x) \leq \mu(0)$ $\forall$ $x \in X$ then

$$\lambda(x.\ a) = \min\{\mu(0), \lambda(x.\ a)\} = (\mu \times \lambda)(0, x.\ a) = (\mu \times \lambda)(0.0, x.\ a) = (\mu \times \lambda)((0,x).(0,a)) \geq \min\{(\mu \times \lambda)(0,x), (\mu \times \lambda)(0,a)\}$$
$$= \min\{\min\{\mu(0), \lambda(x)\}, \min\{\mu(0), \lambda(a)\}\} = \min\{\lambda(x), \lambda(a)\ \} .$$

Now,

$$\lambda(x*z) = \min\{\mu(0), \lambda(x*z)\} = (\mu \times \lambda)(0, x*z) = (\mu \times \lambda)(0*0, x*z) = (\mu \times \lambda)((0,x)*(0,z))$$
$$\geq \min\{(\mu \times \lambda)[(0,x)*((0,y)*(0,z))], (\mu \times \lambda)(0,y)\} = \min\{(\mu \times \lambda)[(0,x)*(0,y*z)], (\mu \times \lambda)(0,y)\}$$
$$= \min(\mu \times \lambda)(0, x*(y*z)), (\mu \times \lambda)(0,y)\} = \min\{\min\{\mu(0), \lambda(x*(y*z))\}, \min\{\mu(0), \lambda(y)\}\}$$
$$= \min\{\lambda(x*(y*z)), \lambda(y)\} .$$

so λ is a fuzzy KS-H-ideal in X .
If $\lambda(x) \leq \mu(0)$ is not satisfied then $\lambda(y) > \mu(0)$ *for some* $y \in X$ and by our assumption ,
$\mu(x) \leq \mu(0)$ *for all* $x \in X$ we have $\lambda(0) \geq \lambda(y) > \mu(0) \geq \mu(x)$ *i.e* $\lambda(0) \geq \mu(x)$ $\forall x \in X$.
therefore $(\mu \times \lambda)(x,0) = \min\{\mu(x), \lambda(0)\} = \mu(x)$ and ,

$$\mu(x.a) = (\mu \times \lambda)(x.a,0)$$
$$= (\mu \times \lambda)(x.a,0.0) = (\mu \times \lambda)((x,0).(a,0)) \geq \min\{(\mu \times \lambda)(x,0), (\mu \times \lambda)(a,0)\} = \min\{\mu(x), \mu(a)\} .$$

so ,

$$\mu(x*z) = (\mu \times \lambda)(x*z,0) = (\mu \times \lambda)(x*z)(0*0) = (\mu \times \lambda)((x,0)*(z,0)) \geq \min\{(\mu \times \lambda)[(x,0)*((y,0)*(z,0))], (\mu \times \lambda)(y,0)\}$$
$$= \min\{(\mu \times \lambda)[(x,0)*(y*z,0)], (\mu \times \lambda)(y,0)\} = \min\{(\mu \times \lambda)((x*(y*z)),0), (\mu \times \lambda)(y,0)\}$$
$$= \min\{\min\{\mu(x*(y*z)), \lambda(0)\}, \min\{\mu(y), \lambda(0)\}\} = \min\{\mu(x*(y*z)), \mu(y)\} .$$

therefore μ is a fuzzy KS-H-ideal in X **.**

References

[1] L.A Zadeh , (1965), "Fuzzy Sets" ,Information Control ,8 ,338-353 .
[2] K.Iseki , (1966) ," An Algebra Related with a Propositional Calculus" , Japan Acad . ,42.
[3] Y.Imai and K.Iseki, (1966), "On Axiom Systems of Propositional Calculi XIV ", Proc.Japan Acad,42 ,19-22 .
[4] Y. B. Jun , S. M. Hong ,J. Meng and X. L. Xin , (1994),"Characterizations of Fuzzy Positive Implicative Ideals in BCK-Algebras" Math . Japon, 40:3,503-507 .

[5] Won Kyun Jeong , (1999)," On Anti Fuzzy Prime Ideal in BCK-Algebras", Journal of the Chungcheong Mathematical Society Volume 12.
[6] Celestin Lele and Congxin Wu Harbin,(2001) , "Generalization of Some Fuzzy Ideal in BCK-Algebras" , Institute of Technology , Department of Mathematics, Harbin 150001,China.
[7] K. H. Kim , (2006), "On Structure of KS-Semigroups" ,Int . Math. Forum , 1, No. 2 , 67-76 .
[8] Young Bae Jun And Seok Zun Song , (2006) "Fuzzy Set Theory Applied To Implicative Ideals in BCK-Algebras " ; Bull . Korean Math. Soc.43, No.3, pp.461- 470.
[9] Williams, D.R, Prince and Husain Shamshad , (2007), "On Fuzzy KS- Semigroup" . International Mathematical Forum ,2, No. 32, 1577-1586 ,.
[10] Joncelyn S.Paradero-Vilela and Mila Cawi, , (2009), " On KS-Semigroup Homomorphism " International Mathematical Forum ,4, no. 23, 1129- 1138.
[11] Saleem Abdullah , Naveed Yaqoob , Bavanari Satyanarayana,S.M.Qurachi , (2012), "Direct product of Intuitionistic Fuzzy H- Ideals of BCK-Algebras , International Journal of Algebra and Statistics 1, 8- 16.
[12] Sajda Kadhum Mohammed,Sundus Najah Jaber ,(2013)," Some Results on Fuzzy subKs-Semigroups ", Journal of kerbala university ,Vol.11,No.3 ,Scientific.